RELATIVE K-STABILITY AND MODIFIED K-ENERGY ON TORIC MANIFOLDS

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Abstract. In this paper, we discuss the relative K-stability and the modified K-energy associated to the Calabi's extremal metric on toric manifolds. We give a sufficient condition in the sense of convex polytopes associated to toric manifolds for both the relative K-stability and the properness of modified K-energy. In particular, our results hold for toric Fano manifolds with vanishing Futaki-invariant. We also verify our results on the toric Fano surfaces.

0. Introduction.

The relation between some geometric stabilities and the existence of Kähler-Einstein metrics, more general, of extremal metrics has been recently studied extensively ([T2], [D1], [D2], [M2], [M3], etc.). The goal is to establish a necessary and sufficient condition for the existence of extremal metrics in the sense of geometric invariants ([Ya], [T2]). More recently, K.S. Donaldson studied a relation between K-stability of Tian-Donaldson and Mabuchi's K-energy on a polarized toric manifold, and for the surface's case he proved that K-energy is bounded from below under the assumption of K-stability ([D2]). So it is a natural problem how to verify K-stability on a polarized Kähler manifold M although we do not know if there exists an extremal metric on M. In this paper, we focus our attention on polarized toric manifolds as in [D2] and give a sufficient condition for K-stability in the sense of integral polytopes associated to polarized toric manifolds. In fact, we can study the relative K-stability (or the relative K-semistability) involved an extremal holomorphic vector field related to an extremal metric on M ([Sz]). This relative K-stability is a generalization of K-stability and removes the obstruction arising from the non-vanishing of Futaki invariant ([FM]).

An n-dimensional polarized toric manifold M is corresponding to an integral polytope P in $\mathbb{R}^n$ which is described by a common set of some half-spaces,

$$\langle l_i, x\rangle < \lambda_i,\ i = 1, ..., d. \tag{0.1}$$

Here l_i are d vectors in $\mathbb{R}^n$ with all components in $\mathbb{Z}$, which satisfy the Delzant condition ([Ab]). Without the loss of generality, we may assume that the original point 0 lies in

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P, so all $\lambda_i > 0$. Let θ_X be a potential function associated to an extremal holomorphic vector field X and a Kähler metric on M. Define

$$\|\theta_X\| = \max_M\{|\theta_X|\}.$$

One can show that $\|\theta_X\|$ is independent of choice of Kähler metrics in the Kähler class ([M1]). Let $\bar{R}$ be the average of scalar curvature of a Kähler metric in the Kähler class. Then we have

Theorem 0.1. *Let M be an n-dimensional polarized toric manifold and X be an extremal vector field on M associated to the polarized Kähler class. Suppose that*

$$\bar{R} + \|\theta_X\| \le \frac{n+1}{\lambda_i}, \ \forall\ i = 1, ..., d. \tag{0.2}$$

Then M is K-stable relative to the $\mathbb{C}^*$-action induced by X for any toric degeneration.

According to [D2], a toric degeneration is a special test configuration induced by a positive rational, piecewise linear function on an integral polytope associated to a polarized toric manifold. We will show that the relative K-semistability for a toric degeneration is a necessary condition for the existence of extremal metrics on such a manifold. In case that M is toric Fano manifold, condition (0.2) in Theorem 0.1 becomes

$$\bar{R} + \|\theta_X\| \le n + 1, \tag{0.2'}$$

since in this case all λ_i are 1. In particular, (0.2') is trivial for toric Fano manifolds with vanishing Futaki-invariant since $\theta_X = 0$, and so Theorem 0.1 is true for these manifolds. We also verify that (0.2') is true on a toric Fano surface (cf. Section 5). It is interesting to ask whether (0.2') is true for higher dimensional toric Fano manifolds or not.

The idea of proof of Theorem 0.1 is to estimate a modified Futaki-invariant related to an extremal metric in the sense of a linear functional on a class of convex functions on P. The late will rely on the structure of an integral polytope associated to a polarized toric manifold (cf. Section 3, 4). Our method here can be also used to study a modified K-energy $\mu(\varphi)$ on a toric Kähler manifold, not only a polarized toric manifold. In the other words, the numbers λ_i in the relation (0.1) need not to be integral. Note that the complex structure on a toric manifold M is uniquely determined by the set $\{l_i\}_{i=1,...,d}$ of d vectors in $\mathbb{R}^n$ and a different set $\{\lambda_i\}_{i=1,...,d}$ of d numbers determines a different Kähler class ([Gui]).

As same as the K-energy, $\mu(\varphi)$ is defined on a Kähler class whose critical point is an extremal metric while the critical point of K-energy is a Kähler metric with constant scalar curvature.

Definition 0.1. *Let ω_g be a Kähler form of Kähler metric g on a compact Kähler manifold M and ω_φ be a Kähler form associated to a potential function φ in Kähler class $[\omega_g]$. Let*

$$I(\varphi) = \frac{1}{V}\int_M \varphi(\omega_g^n - \omega_\varphi^n),$$

where $V = \int_M \omega_g^n$. $\mu(\varphi)$ is called proper associated to a subgroup G of the automorphisms group $Aut(M)$ in Kähler class $[\omega_g]$ if there is a continuous function $p(t)$ in $\mathbb{R}$ with the property

$$\lim_{t\to+\infty} p(t) = +\infty,$$

such that

$$\mu(\varphi) \geq c \inf_{\sigma\in G} p(I(\varphi_\sigma)) - C$$

for some uniform positive constants c and C independent of φ, where φ_σ is defined by

$$\omega_g + \sqrt{-1}\partial\bar{\partial}\varphi_\sigma = \sigma^*(\omega_g + \sqrt{-1}\partial\bar{\partial}\varphi).$$

In the following we give a sufficient condition for the properness of $\mu(\varphi)$ on a toric Kähler manifold.

Theorem 0.2. *Let M be a toric Kähler manifold associated to a convex polytope P. Let G_0 be a maximal compact subgroup of torus actions group T on M. Suppose that condition (0.2) is satisfied. Then $\mu(\varphi)$ is proper associated to subgroup T in the space of G_0-invariant Kähler metrics. In fact, there exists numbers $\delta > 0$ and C such that for any G_0-invariant potential functions associated to the Kähler class, it holds*

$$\mu(\varphi) \geq \delta \inf_{\tau\in T} I(\varphi_\tau) - C_\delta. \tag{0.3}$$

Since condition (0.2) in Theorem 0.1 is true on a toric Fano surface, we see that inequality (0.3) holds on such a manifold. It is known that toric Fano surfaces are classified into five different types, i.e., $\mathbb{C}P^2$, $\mathbb{C}P^1\times\mathbb{C}P^1$ and $\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}$ $(l=1,2,3)$ and there exists a Kähler-Einstein on each one except $\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}$ $(l=1,2)$ ([T1]). On the other hand, it is still unknown whether there exists an extremal metric on $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ while the extremal metric was constructed on $\mathbb{C}P^2\#1\overline{\mathbb{C}P^2}$ by using the ODE method ([Ca], [Ab]). We hope that our result about the properness of modified K-energy will be useful in the study of the existence problem of extremal metrics on $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ ([LS]). In fact, one may guess that there always exists an extremal metric on the Kähler class $2\pi c_1(M)$ on a toric Fano manifold. We note that the existence problem was solved recently in [WZ] when $\theta_X = 0$, which is equal to the existence of Kähler-Einstein metrics with positive scalar curvature.

When the Futaki invariant vanishes on the Kähler class, the extremal vector holomorphic field should be zero. Then condition (0.2) becomes

$$\bar{R} \leq \frac{n+1}{\lambda_i}, \text{ for all, } i = 1, ..., d. \tag{0.2''}$$

Condition (0.2") seems not so hard to satisfy. For instants, we construct a kind of of Kähler classes on the toric manifold $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, which satisfy the condition (0.2") (Section 6).

Since $\bar{R}$ can be computed as some integrals in the sense of convex polytope P (cf. Section 4), as a corollary of Theorem 0.2, we have

Corollary 0.1. *Let M be an n-dimensional toric Kähler manifold with vanishing Futaki invariant. Suppose*

$$\max_{i=1,\ldots,d}\{\frac{\lambda_i}{\sum_{j=1}^d \int_{P_j} dx}\sum_{j=1}^d \frac{1}{\lambda_j}\int_{P_j} dx\} \le \frac{n+1}{n}, \tag{0.4}$$

where $\{P_j\}_{j=1}^d$ is a union of cones with $(n-1)$-dimensional faces E_j on ∂P as bases and vertex at 0. Then K-energy $\mu(\varphi)$ satisfies (0.3). In particular, K-energy $\mu(\varphi)$ is bounded from below in the space of G_0-invariant potential functions.

Finally, we remark that the properness of K-energy is a necessary and sufficient condition for the existence of Kähler-Einstein metrics with positive scalar curvature, and in case that M is a Kähler-Einstein manifold with positive first Chern class, (0.3) implies

$$\mu(\varphi) \ge \delta I(\varphi) - C_\delta, \tag{0.3'}$$

for any $\varphi \in \Lambda_1^{\perp}(M, g_{KE})$, where $\Lambda_1(M, g_{KE})$ denotes the first non-zero eigenfunctions space for the Lapalace operator associated to a Kähler-Einstein metric g_{KE} (cf. Section 4). The existence of such an inequality was conjectured by Tian for any Kähler-Einstein manifold with positive first Chern class ([T2]).

The organization of this paper is as follow. In Section 1, we derive a Donaldson's version for the modified K-energy on toric manifolds. Then in Section 2, we give an analytic criterion for the properness of modified K-energy. In Section 3, we recall the relative K-stability (relative K-semistability) introduced in [Sz] and compute the modified Futaki invariant for a kind of toric degenerations. Both Main Theorem 0.1 and 0.2 will be proved in Section 4. In Section 5, we estimate the potential function of an extremal vector field on $\mathbb{C}P^2\#\overline{1\mathbb{C}P^2}$ and $\mathbb{C}P^2\#\overline{2\mathbb{C}P^2}$ respectively, then verify condition (0.2) for toric Fano surfaces. In Section 6, we construct a kind of of Kähler classes on the toric manifold $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, which satisfy the condition (0.2").

1. A Donaldson's version of $\mu(\varphi)$.

In this section, we review some basic materials of toric differential geometry, then apply them to the modified K-energy $\mu(\varphi)$.

Let (M, g) be a compact Kähler manifold with dimension n. Then Kähler form ω_g of g is given by

$$\omega_g = \sqrt{-1}\sum_{i,j=1}^n g_{i\overline{j}} dz^i \wedge d\overline{z}^j$$

in local coordinates $(z_1, ..., z_n)$, where $g_{i\overline{j}}$ are components of metric g. The Kähler class $[\omega_g]$ of ω_g can be represented by a set of potential functions as follow

$$\mathcal{M} = \{\varphi \in C^\infty(M)|\ \omega_\varphi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0\}.$$

According to [Ca], a Kähler metric ω_φ in the Kähler class $[\omega_g]$ is called extremal if

$$R(\omega_\varphi) = \overline{R} + \theta_X(\omega_\varphi)$$

for some Hamiltonian holomorphic vector field X on M, where $R(\omega_\varphi)$ is the scalar curvature of ω_φ, $\overline{R} = \frac{1}{V}\int_M R(\omega_g)\omega_g^n$, $V = \int_M \omega_g^n$ and $\theta_X(\omega_\varphi)$ denotes the potential function of X associated to the metric ω_φ, which is defined by

$$\begin{cases} i_X\omega_\varphi = \sqrt{-1}\partial\theta_X(\omega_\varphi), \\ \int_M \theta_X(\omega_\varphi)\omega_\varphi^n = 0. \end{cases} \tag{1.1}$$

By [FM], such an X, usually called extremal is uniquely determined by the Futaki invariant $F(\cdot)$. When the Futaki invariant vanishes, then $X = 0$ and an extremal metric becomes one with constant scalar curvature. It is well-known that a Kähler metric with constant scalar curvature is a critical point of the Mabuchi's K-energy. In the case of extremal metrics, one can modify the K-energy to be

$$\mu(\varphi) = -\frac{1}{V}\int_0^1\int_M \dot{\psi}_t[R(\omega_{\psi_t}) - \overline{R} - \theta_X(\psi_t)]\omega_{\psi_t}^n \wedge dt,$$

where $\psi_t (0 \le t \le 1)$ is a path connecting 0 to φ in $\mathcal{M}$. It can be showed that the functional $\mu(\varphi)$ is well-defined, i.e., it is independent of the choice of path ψ_t ([Gua]). Thus φ is a critical point of $\mu(\cdot)$ iff the corresponding metric ω_φ is extremal.

Now we assume that M is an n-dimensional toric Kähler manifold and g is a $G_0 \cong (S^1)^n$-invariant Kähler metric in the Kähler class, where G_0 is a maximal compact subgroup of torus actions group T on M. Then under an affine logarithm coordinates system $(w_1, ..., w_n)$, its Kähler form ω_g is determined by a convex function ψ_0 on $\mathbb{R}^n$, namely

$$\omega_g = \sqrt{-1}\partial\bar{\partial}\psi_0$$

is defined on the open dense orbit T. Denote $D\psi_0$ to be a gradient map (moment map) associated to T. Then the image of $D\psi_0$ is a convex polytope P in $\mathbb{R}^n$. By using the Legendre transformation $y = (D\psi_0)^{-1}(x)$, we see that the function (Legendre function) defined by

$$u_0(x) = \langle y, D\psi_0(y)\rangle - \psi_0(y) = \langle y(x), x\rangle - \psi_0(y(x))$$

is convex on P. In general, for any G_0-invariant potential function φ associated to the Kähler class $[\omega_g]$, one gets a convex function $u(x)$ on P by using the above relation while ψ_0 is replaced by $\psi_0 + \varphi$. Set

$$\mathcal{C} = \{u = u_0 + v |\ u \text{ is a convex function in } \ P,\ v \in C^\infty(\bar{P})\}.$$

It was showed in [Ab] that functions in $\mathcal{C}$ are corresponding to G_0-invariant functions in $\mathcal{M}$ (whose set is denoted by $\mathcal{M}_{G_0}$) by one-to-one.

The convex polytope P is described by a common set of some half-spaces,

$$\langle l_i, x\rangle < \lambda_i,\ i = 1, ..., d, \tag{1.2}$$

where l_i are d-vectors in $\mathbb{R}^n$ with all components in $\mathbb{Z}$, which satisfy the Delzant condition. Conversely, given a convex polytope P in $\mathbb{R}^n$ as above, one can construct a G_0-invariant Kähler metric on M ([Gui]). Without the loss of generality, we may assume that the

original point 0 lies in P, so all $\lambda_i > 0$. Let $d\sigma_0$ be the Lebesgue measure on the boundary ∂P and ν be the outer normal vector field on ∂P. Let $d\sigma = \lambda_i^{-1}(\nu, x)d\sigma_0$ on the face $\langle l_i, x\rangle = \lambda_i$ of P. It is clear that (ν, x) is constant on each face. According to [D2], if $u \in \mathcal{C}$ corresponds to $\varphi \in \mathcal{M}_{G_0}$, then

$$-\frac{1}{2^n n!}\int_0^1 \int_M \dot{\psi}_t R(\omega_{\psi_t})\omega_{\psi_t}^n dt = (2\pi)^n\left(-\int_P \log(\det D^2 u)dx + \int_{\partial P} u d\sigma\right).$$

By using a relation

$$\dot{\psi}_t = -\dot{u_{\psi_t}},$$

we get

$$\mu(\varphi) = \frac{2^n n!(2\pi)^n}{V}\mathcal{F}(u),$$

where

$$\mathcal{F}(u) = -\int_P \log(\det D^2 u)dx + \int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X)u dx. \tag{1.3}$$

Lemma 1.1.

$$\theta_X = \sum_{i=1}^n a_i(x_i + c_i),$$

where $2n$-constants a_i and c_i are determined uniquely by $2n$-equations,

$$\frac{vol(P)}{vol(M)}F(\frac{\partial}{\partial y_i}) = -\int_P (\sum_{j=1}^n a_j(x_j + c_j))(x_i + c_i)dx,\ i = 1, ..., n, \tag{1.4}$$

$$\int_P (x_i + c_i)dx = 0,\ i = 1, ..., n. \tag{1.5}$$

Proof. According to [FM], an extremal holomorphic field X should be belonged to the center of a reductive Lie subalgebra of Lie algebra consisting of holomorphic vector fields on M. In particular, X is an element of an Abelian Lie subalgebra. So under the affine coordinates system, we see

$$X \in \mathrm{span}\{\frac{\partial}{\partial y_i},\ i = 1, ..., n\}.$$

Note that

$$\theta_{\frac{\partial}{\partial y_i}} = x_i + c_i \tag{1.6}$$

for each $\frac{\partial}{\partial y_i}$, where constants c_i are uniquely determined by (1.5) according to (1.1). Thus

$$\theta_X = \sum_{i=1}^n a_i(x_i + c_i) \tag{1.7}$$

for some constants a_i. Since θ_X satisfies ([FM]),

$$\frac{\mathrm{vol}(P)}{\mathrm{vol}(M)}F(\frac{\partial}{\partial y_i}) = -\int_P \theta_X \theta_{\frac{\partial}{\partial y_i}} dx,\ i = 1, ..., n, \tag{1.8}$$

then by relation (1.6) and (1.7), we get (1.4). $\square$

Lemma 1.2. *Let u be an affine linear function on P. Then*

$$L(u) = \int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X) u dx = 0.$$

Proof. First note ([D2])

$$\bar{R} = \frac{\int_P dx}{\int_{\partial P} d\sigma}. \tag{1.9}$$

Then for any constant c, we have $L(c) = 0$. Thus we suffice to prove

$$L(x_i) = 0,\ i = 1, ..., n.$$

Let ρ_t^i be a one-parameter subgroup on M generated by $\mathrm{re}(\frac{\partial}{\partial y_i})$. Then

$$\rho_t^i(y) = \begin{cases} y_j,\ j \neq i \\ y_j + t,\ j = i. \end{cases}$$

Thus $(\varphi + \psi_0)_{\rho_t^i}(y) = (\varphi + \psi_0)(\rho_t^i(y))$ and

$$u_{\rho_t^i}(x) = u - t x_i,$$

where $(\varphi + \psi_0)_{\rho_t^i}$ is a convex function in $\mathbb{R}^n$ induced by actions ρ_t^i as follow,

$$\sqrt{-1}\partial\overline{\partial}(\varphi + \psi_0)_{\rho_t^i} = (\rho_t^i)^* \omega_\varphi,$$

and $u_{\rho_t^i}$ is the Legendre function of $(\varphi + \psi_0)_{\rho_t^i}$. Since

$$\mathrm{vol}(M) \frac{\mu(\varphi_{\rho_t^i})}{dt}|_{t=0} = F(\frac{\partial}{\partial y_i}) + \int_M \theta_X \theta_{\frac{\partial}{\partial y_i}} \omega_g^n = 0,$$

we get

$$2^n n! (2\pi)^n \frac{\mathcal{F}(u_{\rho_t^i})}{dt}|_{t=0} = -L(x_i) = 0.$$

□

By Lemma 1.2, we see that the functional $\mathcal{F}(u)$ is invariant if u is replaced by adding an affine linear function. For this reason, we normalize u as follows. Let $p \in P$ and set

$$\tilde{\mathcal{C}} = \{u \in \mathcal{C} | \inf_P u = u(p) = 0\}.$$

Then for any $u_\varphi \in \mathcal{C}$ corresponding to $\varphi \in \mathcal{M}_{G_0}$, one can normalize u_φ by

$$\tilde{u}_\varphi = u_\varphi - (\langle Du_\varphi(p), x - p\rangle + u_\varphi(p))$$

so that $\tilde{u}_\varphi = u_{\tilde{\varphi}} \in \tilde{\mathcal{C}}$ corresponds to a Kähler potential function $\tilde{\varphi} \in \mathcal{M}_{G_0}$ which satisfies

$$D(\tilde{\varphi} + \psi_0)(0) = p \text{ and } (\tilde{\varphi} + \psi_0)(0) = 0. \tag{1.10}$$

In fact, $\tilde{\varphi}$ can be uniquely determined by using the affine coordinates transformation $y \to y + y_0$ as follow,

$$\tilde{\varphi}(y) = (\varphi + \psi_0)(y + y_0) - \psi_0(y) - (\varphi + \psi_0)(y_0).$$

2 An analytic criterion for the properness of $\mu(\varphi)$.

In this section, we give an analytic criterion for the properness of the modified K-energy $\mu(\varphi)$. We need to recall the Aubin's functional ([Au]),

$$J(\varphi)=\frac{1}{V}\int_0^1\int_M \dot{\psi}_t(\omega_g^n-\omega_{\psi_t}^n)\wedge dt,\ \forall\ \varphi\in\mathcal{M},$$

where $\psi_t(0\le t\le 1)$ is a path in $\mathcal{M}$ connecting 0 to φ. Note $\dot{\psi}_t=-\dot{u}_t$, where u_t are Legendre functions associated to potential functions ψ_t. Then

$$J(\varphi)=\frac{1}{V}\int_M \varphi\omega_g^n+H(u_\varphi)-H(u_0), \tag{2.1}$$

where

$$H(u)=\frac{1}{Vol(P)}\int_P udx,\ \forall\ u\in\mathcal{C}.$$

Lemma 2.1. *There exists $C>0$ such that*

$$|J(\tilde{\varphi})-H(u_{\tilde{\varphi}})|\le C,\ \forall\ \varphi\in\mathcal{M}_{G_0},$$

where $\tilde{\varphi}$ is as defined in (1.10).

Proof. By (2.1), we have

$$J(\tilde{\varphi})-H(u_{\tilde{\varphi}})=\frac{1}{V}\int_M\tilde{\varphi}\omega_g^n-H(u_0).$$

We claim that

$$|\frac{1}{V}\int_M\tilde{\varphi}\omega_g^n|\le C$$

for some uniform constant C.

Applying the Green's formula to the potential function $\tilde{\varphi}$, one sees that there exists a constant C_0 such that

$$\frac{1}{V}\int_M\tilde{\varphi}\omega_g^n\ge\sup\{\tilde{\varphi}\}-C_0. \tag{2.2}$$

Set

$$\Omega_N=\{\xi\in M|\ \tilde{\varphi}(\xi)\le\sup_{\mathbb{R}^n}\{\tilde{\varphi}\}-N\}.$$

Then

$$\begin{aligned}
&\frac{1}{V}\int_M\tilde{\varphi}\omega_g^n\\
&=\frac{1}{V}\int_{M\cap\Omega_N}\tilde{\varphi}\omega_g^n+\frac{1}{V}\int_{M\setminus\Omega_N}\tilde{\varphi}\omega_g^n\\
&\le\frac{1}{V}[(\sup_{\mathbb{R}^n}\{\tilde{\varphi}\}-N)Vol(M\cap\Omega_N)+\sup_{\mathbb{R}^n}\{\tilde{\varphi}\}Vol(M\setminus\Omega_N)]\\
&=\sup_{\mathbb{R}^n}\{\tilde{\varphi}\}-\frac{NVol(M\cap\Omega_N)}{Vol(M)}.
\end{aligned}$$

It follows

$$Vol(M\cap\Omega_N)\le\frac{C_0Vol(M)}{N}=\frac{C_0V}{N}\to 0, \tag{2.3}$$

as $N\to\infty$. On the other hand, by the second relation in (1.10), we have

$$\tilde\varphi(0)=-\psi_0(0).$$

Then

$$\tilde\varphi(x)\le\varphi(0)-2r\sup\{|p|:\ p\in P\}\le C(r),\ \forall\ x\in B_r(0),$$

where $C(r)$ depends only on the radius r of ball $B_r(0)$ centered at the original. Since the volume of domain $B_1(0)\times(2\pi)^n$ associated the metric $\omega_g=\sqrt{-1}\partial\overline{\partial}\psi_0$ is bigger than some number $\epsilon>0$, by (2.3), it is easy to see that there is at least a point $x_0\in B_1(0)$ such that

$$\tilde\varphi(x_0)\ge\sup_{\mathbb{R}^n}\tilde\varphi-N$$

as N is sufficiently large. Thus

$$\sup_{\mathbb{R}^n}\tilde\varphi\le N+C(1),$$

and consequently

$$\frac{1}{V}\int_M\tilde\varphi\omega_g^n\le N+C(1).$$

By (2.2), we also get

$$\frac{1}{V}\int_M\tilde\varphi\omega_g^n\ge\tilde\varphi(0)-C_0=-\psi(0)-C_0.$$

Therefore the claim is true and Lemma is proved. $\square$

As in [D2], one can extend the definition space $\mathcal{C}$ of functional $\mathcal{F}(u)$ to

$$\mathcal{C}_\infty=\{u\in C^\infty(P)\cap C(\overline{P})|\ u \text{ is convex in } P\}$$

and one can show that

$$\inf_{u\in\mathcal{C}}\mathcal{F}(u)=\inf_{u\in\mathcal{C}_\infty}\mathcal{F}(u).$$

Note that the minimal point of $\mathcal{F}(u)$ is unique by using the convexity of $-$logdet and it satisfies the Euler-Langrage equation,

$$\sum_{i,j=1}^n u_{ij}^{ij}=-(\overline{R}+\theta_X),$$

where $(u^{ij})=(u_{ij})^{-1}$ and u_{kl}^{ij} denote the second derivatives of u^{ij}. In fact, the above statement also holds for more general functional $\mathcal{F}(u)$ while the function $\overline{R}+\theta_X$ in (1.3) is replaced by a smooth one in $\overline{P}$.

The following result is due to [D2].

Lemma 2.2. *Suppose that there exists a $\lambda > 0$ such that*

$$L(u) \geq \lambda \int_{\partial P} u d\sigma, \tag{2.4}$$

for any normalized function $u \in \mathcal{C}_\infty$. Then there exists a $\delta > 0$ depending only on λ such that

$$\mathcal{F}(u) \geq \delta H(u) - C_\delta \tag{2.5}$$

for any normalized function $u \in \mathcal{C}_\infty$.

Proof. Choose a function v_0 in $\mathcal{C}$ and define a smooth function A in $\overline{P}$ by

$$v_{0ij}^{ij} = -(\overline{R} + A).$$

Let $\mathcal{F}'(u)$ be a modified functional of $\mathcal{F}(u)$ while the part of linear functional $L(u)$ of $\mathcal{F}(u)$ is replaced by

$$L'(u) = \int_{\partial P} u d\sigma - \int_P (\overline{R} + A) u dx.$$

Then

$$\mathcal{F}'(u) \geq \mathcal{F}'(v_0) = -C_0, \ \forall \ u \in \mathcal{C}_\infty. \tag{2.6}$$

We compute the difference between the linear parts $L'(u)$ and $L(u)$. Pick a $\delta > 0$. Note that

$$\int_P u dy \leq C \int_{\partial P} u d\sigma$$

because of the convexity of $u \geq 0$. Then by the condition (2.4), we have

$$\begin{aligned}
|L(u) - L'(u)| &= \left| \int_P \theta_X u dx - \int_P A u dx \right| \\
&\leq C_1 \int_P u dx \\
&= C_1 \left[(1+\delta) \int_P u dx - C_1 \delta \int_P u dx \right] \\
&\leq C'_\delta \int_{\partial P} u d\sigma - C_1 \delta \int_P u dx \\
&\leq C'_{\delta,\lambda} L(u) - C_1 \delta \int_P u dx.
\end{aligned}$$

It follows

$$(C'_{\delta,\lambda} + 1) L(u) \geq L'(u) + C_1 \delta \int_P u dx.$$

Thus by (2.6), we get

$$\begin{aligned}\mathcal{F}(u) &= -\int_P \log(\det(u_{ij}))dx + L(u)\\ &\geq -\int_P \log(\det(u_{ij}))dx + \frac{L'(u)}{C'_{\delta,\lambda}+1} + \frac{C_1\delta}{C'_{\delta,\lambda}+1}\int_P udx\\ &= \mathcal{F}'(\frac{u}{C'_{\delta,\lambda}+1}) + \frac{C_1\delta}{C'_{\delta,\lambda}+1}\int_P udx - n\log(C'_{\delta,\lambda}+1)\\ &\geq \frac{C_1\mathrm{Vol}(P)\delta}{C'_{\delta,\lambda}+1}H(u) - C'_0.\end{aligned}$$

Replacing $\frac{C_1\mathrm{Vol}(P)\delta}{C'_{\delta,\lambda}+1}$ by δ, we obtain (2.5). $\square$

Proposition 2.1. *Suppose that (2.4) is satisfied. Then there exists a number $\delta > 0$ such that for any G_0-invariant $\varphi \in \mathcal{M}_{G_0}$ it holds*

$$\mu(\varphi) \geq \delta \inf_{\tau\in T^n} I(\varphi_\tau) - C_\delta, \tag{2.7}$$

where

$$\omega_g + \sqrt{-1}\partial\bar\partial\varphi_\tau = \tau^*(\omega_g + \sqrt{-1}\partial\bar\partial\varphi). \tag{2.8}$$

In particular, $\mu(\varphi)$ is bounded from below in $\mathcal{M}_{G_0}$.

Proof. Let $\varphi \in \mathcal{M}_{G_0}$. Then there exists a $\sigma \in T$ such that the Legendre function u_{φ_σ} associated to φ_σ is belonged to $\tilde{\mathcal{C}}$. By Lemma 2.2, we see that

$$\mu(\varphi_\sigma) \geq \delta H(\varphi_\sigma) - C_\delta.$$

Note that

$$\mu(\varphi) = \mu(\varphi_\sigma).$$

Thus by Lemma 2.1, we get

$$\begin{aligned}\mu(\varphi) = \mu(\varphi_\sigma) &\geq \delta J(\varphi_\sigma) - C'_\delta \geq \delta \inf_{\tau\in T^n} J(\varphi_\tau) - C'_\delta\\ &\geq \frac{\delta}{n}\inf_{\tau\in T^n} I(\varphi_\tau) - C'_\delta.\end{aligned}$$

Here at the last inequality we used the fact ([Au]),

$$J(\varphi) \geq \frac{1}{n} I(\varphi),\ \forall\ \varphi \in \mathcal{M}.$$

The proposition is proved. $\square$

3. Computation of the modified Futaki-invariant.

In this section, we recall the notation of relative K-stability introduced in [Sz] and compute the modified Futaki invariant for a test configuration in the sense of Donaldson ([D2]). We assume that (M, L) is an n-dimensional polarized toric manifold and g is a $G_0 \cong (S^1)^n$-invariant Kähler metric with its Kähler form $\omega_g \in 2\pi c_1(L)$ on M , where L is a positive holomorphic line bundle on M. In the other words, the corresponding convex polytope P induced by the moment map is integral (cf. Section 2).

Definition 3.1 ([Do], [Sz]). *A test configuration for a polarized variety (M, L) of exponent r consists of a $\mathbb{C}^*-$equivariant flat family of schemes $\pi : \mathcal{W} \longrightarrow \mathbb{C}$ (where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication) and a $\mathbb{C}^*-$equivariant ample line bundle $\mathcal{L}$ on $\mathcal{W}$. We require that the fibres $(\mathcal{W}_t, \mathcal{L}|_{\mathcal{W}_t})$ are isomorphic to (M, L^r) for any $t \neq 0$. A test configuration is called trivial if $\mathcal{W} = M \times \mathbb{C}$ is a product.*

If (M, L) is equipped with a $\mathbb{C}^*-$action β, we say that a test configuration is compatible with β, if there is a $\mathbb{C}^*-$action $\tilde{\beta}$ on $(\mathcal{W}, \mathcal{L})$ such that $\pi : \mathcal{W} \longrightarrow \mathbb{C}$ is $\tilde{\beta}$ equivariant with trivial $\mathbb{C}^*-$action on $\mathbb{C}$ and the restriction of $\tilde{\beta}$ to $(\mathcal{W}_t, \mathcal{L}|_{\mathcal{W}_t})$ for nonzero t coincides with that of β on (M, L^r) under the isomorphism.

Note that a $\mathbb{C}^*$-action on $\mathcal{W}$ induces a $\mathbb{C}^*$-action on the central fibre $M_0 = \pi^{-1}(0)$ and the restricted line bundle $\mathcal{L}|_{M_0}$. We denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the induced $\mathbb{C}^*-$actions of α and β on $(M_0, \mathcal{L}|_{M_0})$, respectively. The relative K-semistability is based on the following modified Futaki invariant on the central fibre,

$$F_{\tilde{\beta}}(\tilde{\alpha}) = F(\tilde{\alpha}) - \frac{(\tilde{\alpha}, \tilde{\beta})}{(\tilde{\beta}, \tilde{\beta})} F(\tilde{\beta}), \tag{3.1}$$

where $F(\tilde{\alpha})$ and $F(\tilde{\beta})$ are generalized Futaki invariants of $\tilde{\alpha}$ and $\tilde{\beta}$ defined in [D2], respectively, and $(\tilde{\alpha}, \tilde{\beta})$ and $(\tilde{\beta}, \tilde{\beta})$ are inner products defined in [Sz] (also to see (3.7) below).

Definition 3.2 ([Sz]). *A polarized variety (M, L) with a $\mathbb{C}^*$-action β is K-semistable relative to β if $F_{\tilde{\beta}}(\cdot) \leq 0$ for any test-configuration compatible with β. It is called relative K-stable in addition that the equality holds if and only if the test-configuration is trivial.*

Now we consider for a polarized toric manifold (M, L) which corresponds to an integral polytope P in $\mathbb{R}^n$. Recall that a piecewise linear (PL) function u on P is a form of

$$u = \max\{u^1, ..., u^r\},$$

where $u^\lambda = \sum a_i^\lambda x_i + c^\lambda,\ \lambda = 1, ..., r$, for some vectors $(a_i^\lambda) \in \mathbb{R}^n$ and some numbers $c^\lambda \in \mathbb{R}$. u is called a rational PL-function if components a_i^λ and numbers c^λ are all rational. For a rational PL function u on P, choose an integer R so that

$$Q = \{(x, t)|\ x \in P, 0 < t < R - u(x)\}$$

is a convex polytope Q in $\mathbb{R}^{n+1}$. Then M_Q corresponds to $(n+1)$-dimensional toric variety and L on M induces a holomorphic line bundle $\mathcal{L}$ on M_Q by using the natural

embedding $i : M \to M_Q$. Decomposing a torus action $T^{n+1}_{\mathbb{C}}$ on M_Q as $T^n_{\mathbb{C}} \times \mathbb{C}^*$ so that $T^n_{\mathbb{C}} \times \{\mathrm{Id}\}$ is isomorphic to the torus action on M, we get a $\mathbb{C}^*-$action α by $\{\mathrm{Id}\} \times \mathbb{C}^*$, and so we define an equivariant map

$$\pi : M_Q \to \mathbb{C}P^1$$

satisfying $\pi^{-1}(\infty) = i(M)$. Then $\mathcal{W} = M_Q \backslash i(M)$ is a test configuration for the pair (M, L), called a toric degeneration ([D2]). This test configuration is compatible to an extremal $\mathbb{C}^*-$action β induced by an extremal holomorphic vector field X on M. In fact, the extremal $\mathbb{C}^*-$action β is isomorphic to a one parameter subgroup of $T^n_{\mathbb{C}} \times \{\mathrm{Id}\}$, which acts on $\mathcal{W}$. Since the action is trivial in the direction of α, it is compatible. To compute the modified Futaki invariant for such a test configuration, we need

Lemma 3.1. *Let $B_{k,P} = Z^n \bigcap k\bar{P}$ for any $k \in Z^+$. Assume P is an integral polytope in R^n and φ, ψ are two positive rational, PL-functions on P. Then*

$$\sum_{I \in B_{k,P}} \varphi(I) = k^n \int_P \varphi dx + \frac{k^{n-1}}{2} \int_{\partial P} \varphi d\sigma + O(k^{n-2}) \tag{3.2}$$

and

$$\sum_{I \in B_{k,P}} \varphi(I)\psi(I) = k^n \int_P \varphi\psi dx + O(k^{n-1}). \tag{3.3}$$

Proof. The relation (3.3) in the lemma is trivial and it suffices to prove (3.2). Let Q be a convex polytope associated to rational, PL-function φ as above and $B_{k,Q} = Z^{n+1} \bigcap k\bar{Q}$. Then it is easy to see that

$$\sum_{I \in B_{k,P}} \varphi(I) = N(B_{k,Q}) - N(B_{k,P}),$$

where $N(B_{k,Q}), N(B_{k,P})$ are the numbers of point, respectively. Applying Proposition 4.1.3 in [D2] for each convex polytope P and Q, one will get (3.2). □

Proposition 3.1. *For the above test configuration induced by a rational PL-function u, we have*

$$F_{\tilde{\beta}}(\tilde{\alpha}) = -\frac{1}{2Vol(P)} \left(\int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X) u dx \right). \tag{3.4}$$

Proof. As in [D2], we consider the space $H^0(\mathcal{W}, \mathcal{L}^k)$ of holomorphic sections over $\mathcal{W}$, which has a basis $\{S_{I,i}\}$, where I is a lattice in $B_{k,P}$ and $0 \le i \le k(R-u)(I)$. By using the exact sequence for large k,

$$0 \longrightarrow H^0(\mathcal{W}, \mathcal{L}^k \otimes \pi^*(\vartheta(-1))) \longrightarrow H^0(\mathcal{W}, \mathcal{L}^k) \longrightarrow H^0(M_0, \mathcal{L}^k) \longrightarrow 0,$$

$H^0(M_0, \mathcal{L}^k)$ has a basis $\{S_{I,k(R-u)(I)}|_{M_0}\}_{I \in B_{k,P}}$. Then by (3.2), one obtains

$$\begin{aligned} d_k = \dim H^0(M, \mathcal{L}^k) &= \dim H^0(M, \mathcal{L}^k) = N(B_{k,P}) \\ &= k^n Vol(P) + \frac{k^{n-1}}{2} Vol(\partial P) + O(k^{n-2}). \end{aligned}$$

By choosing a suitable coordinates system, we may assume that

$$\int_P x_i dx = 0.$$

Then $\theta_X = \langle \theta, x \rangle$ for some vector θ in $\mathbb{R}^n$ and the one parameter subgroup β induced by X in $T_{\mathbb{C}}^{n+1}$ is a form of

$$(e^{\theta_1 z}, ..., e^{\theta_n z}, 1),\ z \in \mathbb{C}^*,$$

which act on $S_{I,i}$ with weight $k\langle \theta, I \rangle$. On the other hand, α in $T_{\mathbb{C}}^{n+1}$ is a form of

$$(1, ..., 1, e^z),$$

which act on $S_{I,i}$ with weight $k(R-u)(I)$. Thus the infinitesimal generators A_k and B_k of $\tilde{\alpha}$ and $\tilde{\beta}$ are $(d_k \times d_k)$ diagonal matrices

$$diag(..., k(R-u)(I), ...)$$

and

$$diag(..., k\langle I, \theta \rangle, ...),$$

respectively. By Lemma 3.1, we get follows,

$$\begin{aligned} Tr(A_k) &= \sum_{I \in B_{k,P}} k(R-u)(I) \\ &= k^{n+1} \int_P (R-u) dx + \frac{k^n}{2} \int_{\partial P} (R-u) d\sigma + O(k^{n-1}), \end{aligned}$$

$$\begin{aligned} Tr(B_k) &= \sum_{I \in B_{k,P}} k\langle \theta, I \rangle \\ &= k^{n+1} \int_P \theta_X dx + \frac{k^n}{2} \int_{\partial P} \theta_X d\sigma + O(k^{n-1}), \end{aligned}$$

$$\begin{aligned} Tr(A_k B_k) &= \sum_{I \in B_{k,P}} k^2 (R-u)(I) \langle \theta, I \rangle \\ &= k^{n+2} \int_P (R-u) \theta_X dx + O(k^{n+1}), \end{aligned}$$

and

$$\begin{aligned} Tr(B_k^2) &= \sum_{I \in B_{k,P}} k^2 \langle \theta, I \rangle^2 \\ &= k^{n+2} \int_P \theta_X^2 dx + O(k^{n+1}). \end{aligned}$$

Hence

$$F(\tilde{\alpha}) = -\frac{1}{2Vol(P)} \left[\int_{\partial P} u d\sigma - \frac{Vol(\partial P)}{Vol(P)} \int_P u dx \right],$$

$$F(\tilde{\beta}) = -\frac{1}{2Vol(P)} \int_P \theta_X^2 dx. \tag{3.5}$$

Recall that

$$\overline{R} = \frac{Vol(\partial P)}{Vol(P)}.$$

Therefore, we get

$$F(\tilde{\alpha}) = -\frac{1}{2Vol(P)} \left[\int_{\partial P} u d\sigma - \bar{R} \int_P u dx \right]. \tag{3.6}$$

By Lemma 1.2, we have

$$L(\theta_X) = 0,$$

which is equal to

$$\int_{\partial P} \theta_X d\sigma - \int_P (\bar{R} + \theta_X)\theta_X dx = 0.$$

Note that

$$\int_P \theta_X dx = 0.$$

It follows

$$\int_{\partial P} \theta_X d\sigma = \int_P \theta_X^2 dx.$$

Thus by the relation ([Sz]),

$$Tr(A_k B_k) - \frac{Tr(A_k)Tr(B_k)}{d_k} = (\tilde{\alpha}, \tilde{\beta}) k^{n+2} + O(k^{n+1}), \tag{3.7}$$

we get

$$(\tilde{\alpha}, \tilde{\beta}) = -\int_P \theta_X u dx. \tag{3.8}$$

Similarly, we have

$$(\tilde{\beta}, \tilde{\beta}) = -\int_P \theta_X^2 dx. \tag{3.9}$$

Substituting (3.5), (3.6), (3.8) and (3.9) into (3.1), we obtain (3.4) □

4. Proof of Theorem 0.1 and 0.2.

In this section, we prove the main theorems in Introduction. As in Section 1, we let P be a convex polytope in $\mathbb{R}^n$ defined by (1.2) which satisfies Delzant condition, and $L(u)$ be the linear functional

$$L(u) = \int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X) u dx$$

defined on the space $\mathcal{C}$. Without the loss of generality, we may assume that P contains the original point 0. Let $\{E_i\}_{i=1}^d$ be a union of $(n-1)$-dimensional faces on ∂P and $\{P_i\}_{i=1}^d$ be a union of cones with bases E_i and the vertex at 0. First we observe

Lemma 4.1.
$$L(u)=\sum_{i=1}^{d}\int_{P_i}\left[\frac{\sum_{j=1}^n x_ju_j}{\lambda_i}+(\frac{n}{\lambda_i}-\bar{R}-\theta_X)u\right]dx. \tag{4.1}$$

Proof. Recall that
$$d\sigma=\frac{(\nu,x)}{\lambda_i}d\sigma_0,\quad \text{on each } E_i,$$
where ν is the outer normal vector field on ∂P. Since
$$(\nu,x)\equiv 0,\quad \text{on each } P_i\setminus E_i,$$
by the Stoke's formula, we have
$$\begin{aligned}\int_{E_i}ud\sigma&=\int_{\partial P_i}u\frac{(\nu,x)}{\lambda_i}d\sigma_0\\&=\int_{P_i}div(\frac{xu}{\lambda_i})dx\\&=\int_{P_i}(\frac{\sum_{j=1}^n x_ju_j}{\lambda_i}+\frac{n}{\lambda_i}u)dx.\end{aligned}$$
Summing these identities, we get
$$\int_{\partial P}ud\sigma=\sum_{i=1}^{d}\int_{P_i}(\frac{\sum_{j=1}^n x_ju_j}{\lambda_i}+\frac{n}{\lambda_i}u)dx.$$
Then (4.1) follows from the above. □

Lemma 4.2. *Let u be a normalized function at the original point* 0*. Then*
$$L(u)\geq\sum_{i=1}^{d}\int_{P_i}(\frac{n+1}{\lambda_i}-\bar{R}-\theta_X)udx. \tag{4.2}$$

Proof. By Lemma 4.1, we have
$$\begin{aligned}L(u)=&\sum_{i=1}^{d}\int_{P_i}\frac{\sum_{j=1}^n x_ju_j-u}{\lambda_i}dx\\&+\sum_{i=1}^{d}\int_{P_i}(\frac{n+1}{\lambda_i}-\bar{R}-\theta_X)udx.\end{aligned}$$
Note that $(\sum_{i=1}^n x_iu_i-u)$ is the Legendre function of u and so it is nonnegative by the the normalized condition. Thus (4.2) is true. □

Note that the functional $L(u)$ can be defined for a PL-function u on P introduced in Section 3 and (4.1) still holds. Now we begin to prove Theorem 0.1 and need

Lemma 4.3. *Let M be a toric Kähler manifold associated to a convex polytope P and X be its corresponding extremal vector field on M. Suppose that for each $i = 1, ..., d$, it holds*

$$\overline{R} + \theta_X \leq \frac{n+1}{\lambda_i}, \ \text{in } P. \tag{4.3}$$

Then for any PL-function u on P, we have

$$L(u) \geq 0. \tag{4.4}$$

Moreover the equality holds if and only if u is an affine linear function.

Proof. Let u be a form of

$$u = \max\{u^1, ..., u^r\},$$

where $u^\lambda = \sum a_i^\lambda x_i + c^\lambda, \ \lambda = 1, ..., r$, for some vectors $(a_i^\lambda) \in \mathbb{R}^n$ and some numbers $c^\lambda \in \mathbb{R}$. By adding an suitable affine linear function so that u is normalized to be $\tilde{u}$ with properties

$$\tilde{u} \geq 0, \ \text{in } P$$

and

$$\tilde{u}(0) = 0.$$

Denote $\tilde{u}^\lambda$ and $\tilde{c}^\lambda$ to be the corresponding linear functions and numbers respectively, as to u. Then it is easy to see that $\tilde{c}^\lambda \leq 0$ for all λ. Dividing P into r pieces $P^1, ..., P^r$ so that each $\tilde{u} = \tilde{u}^\lambda$ is defined on each P^λ, then by Lemma 4.1 and the condition (4.3), we have

$$\begin{aligned} L(u) = L(\tilde{u}) &= \sum_{i=1}^d \int_{P_i} \left[\frac{\sum_{j=1}^n x_j \tilde{u}_j}{\lambda_i} + (\frac{n}{\lambda_i} - \bar{R} - \theta_X)\tilde{u} \right] dx \\ &= \sum_{i=1}^d \sum_{\lambda=1}^r \int_{P_i \cap P^\lambda} \left[\frac{\sum_{j=1}^n x_j \tilde{u}_j}{\lambda_i} + (\frac{n}{\lambda_i} - \bar{R} - \theta_X)\tilde{u} \right] dx \\ &= \sum_{i=1}^d \sum_{\lambda=1}^r \int_{P_i \cap P^\lambda} \left[\frac{-\tilde{c}^\lambda}{\lambda_i} + (\frac{n+1}{\lambda_i} - \bar{R} - \theta_X)\tilde{u}^\lambda \right] dx \\ &= \sum_{i=1}^d \sum_{\lambda=1}^r \int_{P_i \cap P^\lambda} (\frac{n+1}{\lambda_i} - \bar{R} - \theta_X)\tilde{u}^\lambda \geq 0. \end{aligned} \tag{4.5}$$

Thus we prove (4.4). Moreover, since each set $\{\frac{n+1}{\lambda_i} - \bar{R} - \theta_X = 0\} \cap P$ lies in a hyperplane in $\mathbb{R}^n$, one sees that the equality in (4.5) holds if and only if $\tilde{u} = 0$, which is equivalent to that u is an affine linear function. □

Proof of Theorem 0.1. Let $u \geq 0$ be a rational PL-function which associates to a toric degeneration on M. Then by Lemma 4.3, we have

$$\begin{aligned} &L(u) > 0, \ \text{if } u \text{ is not an affine linear function, or} \\ &L(u) = 0, \ \text{if } u \text{ is an affine linear function.} \end{aligned}$$

The later implies that $u \equiv 0$ since $u \geq 0$, and consequently, the corresponding toric degeneration is trivial. Thus by Proposition 3.1, we have

$$F_{\tilde{\beta}}(\tilde{\alpha}) = -\frac{1}{2Vol(P)}L(u) < 0,$$

if the toric degeneration is not trivial, where $F_{\tilde{\beta}}(\tilde{\alpha})$ denotes the modified Futaki invariant on the central fibre associated to a $\mathbb{C}^*$-action induced by the toric degeneration. The theorem is proved. □

Next we show that the K-semistability is a necessary condition for the existence of extremal metrics on a polarized toric manifold. We need

Lemma 4.4. *If $\mu(\varphi)$ is bounded from below in $\mathcal{M}_{G_0}$, then*

$$L(u) \geq 0, \forall\ u \in \mathcal{C}_\infty. \tag{4.6}$$

Proof. Suppose that there exists f in $\mathcal{C}_\infty$ such that

$$L(f) < 0.$$

Then we choose a function u in $\mathcal{C}$ and consider a sequence of $u_k = u + kf \in \mathcal{C}_\infty$. By the convexity of $\mathcal{F}(u)$, we have

$$\mathcal{F}(u_k) \leq \mathcal{F}(u) + kL(f) \to -\infty$$

as $k \to \infty$. This is impossible since $\mu(\varphi)$ is bounded from below. Thus (4.6) is true. □

Proposition 4.1. *Let M be a polarized toric manifold which admits an extremal metric. Then M is K-semistable relative to the $\mathbb{C}^*$-action induced by X for any toric degeneration.*

Proof. Note that the existence of extremal metrics implies that $\mu(\varphi)$ is bounded from below in $\mathcal{M}_{G_0}$. Then by Lemma 4.4, (4.6) is true. Now suppose that the proposition is not true. Then by Proposition 3.1, there exists a positive PL-function u such that $L(u) < 0$. By making a small perturbation to u, one will get a smooth function u' in $\mathcal{C}_\infty$ such that $L(u') < 0$. But this is impossible by (4.6). Thus the proposition is true. □

Proof of Theorem 0.2. By Proposition 2.1, it suffices to prove that the condition (2.4) holds for any normalized function $u \in \mathcal{C}_\infty$ with

$$\int_{\partial P} u d\sigma = 1. \tag{4.7}$$

As in Lemma 4.2, we may assume that P contains the origin 0 and u is normalized at 0. Moreover, we have

$$L(u) \geq 0,$$

otherwise, one can find a PL-function u' such that

$$L(u') < 0,$$

which is impossible according to Lemma 4.3. Thus by the contradiction, if (2.4) is not true, then there is a sequence of normalized functions $\{u^{(k)}\}$ in $\mathcal{C}_\infty$ such that

$$\int_{\partial P} u^{(k)} d\sigma = 1 \tag{4.8}$$

and

$$L(u^{(k)}) \longrightarrow 0, \text{as } k \longrightarrow \infty. \tag{4.9}$$

By Lemma 4.3 and (4.9), we have

$$L(u^{(k)}) \geq \sum_{i=1}^{d} \int_{P_i} (\frac{n+1}{\lambda_i} - \bar{R} - \theta_X) u^{(k)} dx \longrightarrow 0.$$

On the other hand, by (4.8), we see that there exists a subsequence (still denoted by $\{u^{(k)}\}$) of $\{u^{(k)}\}$, which converges locally uniformly to a normalized convex function $u_\infty \geq 0$ on P. It follows

$$\begin{aligned} &\sum_{i=1}^{d} \lim_k \int_{P_i} (\frac{n+1}{\lambda_i} - \bar{R} - \theta_X) u^{(k)} dx \\ &= \sum_{i=1}^{d} \int_{P_i} (\frac{n+1}{\lambda_i} - \bar{R} - \theta_X) u_\infty dx = 0. \end{aligned} \tag{4.10}$$

We claim that

$$u_\infty \equiv 0, \text{ on } P. \tag{4.11}$$

If (4.11) is not true, then there is an open set U of P such that $u_\infty > 0$. Since each set $\{\frac{n+1}{\lambda_i} - \bar{R} - \theta_X = 0\} \cap P$ lies in a hyperplane in $\mathbb{R}^n$, we get

$$\begin{aligned} &\sum_{i=1}^{d} \int_{P_i} (\frac{n+1}{\lambda_i} - \bar{R} - \theta_X) u_\infty dx \\ &\geq \sum_{i=1}^{d} \int_{U \cap P_i} (\frac{n+1}{\lambda_i} - \bar{R} - \theta_X) u_\infty dx > 0, \end{aligned}$$

which is a contradiction to (4.10). Thus (4.11) is true.

By the assumption conditions (4.8) and (4.9), one sees

$$\begin{aligned} &\lim_k \int_P (\overline{R} + \theta_X) u^k dx \\ &= \lim_k \int_{\partial P} u^k d\sigma = 1. \end{aligned}$$

On the other hand, similarly to (4.10), we have

$$\begin{aligned}&\lim_k \int_P (\overline{R}+\theta_X)u^k dx\\&=\int_P (\overline{R}+\theta_X)u_\infty dx = 0.\end{aligned}$$

Thus we get a contradiction from the above. The contradiction shows that (2.4) is true. The theorem is proved. □

Proof of Corollary 0.1. By (1.9) in Section 1 and Lemma 4.1, we see

$$\int_{\partial P} d\sigma = n\sum_{i=1}^d \int_{P_i} \frac{1}{\lambda_i} dx.$$

Then condition (0.2”) in Corollary 0.1 implies condition (0.2) since $\theta_X = 0$. Thus the corollary follows from Theorem 0.2. □

For the case of toric Fano surfaces, condition (2.4) can be proved in another way. Recall that a simple PL-function is a form of

$$u = \max\{0, \sum a_i x_i + c\}$$

for some vector $(a_i) \in \mathbb{R}^n$ and number $c \in \mathbb{R}$. We call the hyperplane $\sum a_i x_i + c = 0$ a crease of u. The following result was proved in [D2] as a special case of Proposition 5.2.2 and 5.3.1 there.

Lemma 4.5. *Let M be a toric surface with*

$$\bar{R}+\theta_X \geq 0. \tag{4.12}$$

Suppose that the condition (2.4) in Section 2 doesn't hold. Then either

1. *there exists a positive rational PL-function u such that $L(u) < 0$, or*
2. *there exists a simple PL-function u with crease intersecting the interior of the convex polytope such that $L(u) = 0$.*

It is known that toric Fano surfaces are classified into five different types, i.e., $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}$ $(l = 1, 2, 3)$ and there exists a Kähler-Einstein on each one except $\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}$ $(l = 1, 2)$ ([T1]). For the last two cases we will verify the condition (4.12) in the next section. Thus combining Lemma 4.3 and 4.5, we see that condition (2.4) is true if (4.3) holds.

The condition (4.3) is also true for toric Fano surfaces (cf. Section 5). Thus as a corollary of Theorem 0.2, we have

Corollary 4.1. *Let M be a toric Fano surface and G_0 be a maximal compact subgroup of torus actions group T on M. Then there exists numbers $\delta > 0$ and C such that for any G_0-invariant potential functions associated to the Kähler class $2\pi c_1(M)$, it holds*

$$\mu(\varphi) \geq \delta \inf_{\tau\in T} I(\varphi_\tau) - C_\delta.$$

According to [WZ], any toric Fano manifold admits a Kähler-Einstein metric iff the Futaki invariant vanishes. Then by Theorem 0.2, we have

Corollary 4.2. *Let M be a toric Fano manifold with the vanishing Futaki invariant. Then there exists $\delta > 0$ and C such that*

$$\mu(\varphi) \geq \delta I(\varphi) - C_\delta \tag{4.13}$$

for any $\varphi \in \Lambda_1^\perp(M, g_{KE}) \cap \mathcal{M}_{G_0}$, where $\Lambda_1(M, g_{KE})$ denotes the first non-zero eigenfunctions space of the Lapalace operator associated to the G_0-invariant Kähler-Einstein metric g_{KE} on M.

Proof. We choose the Kähler-Einstein metric g_{KE} on M as an initial metric in the definition of potential functions space $\mathcal{M}$ and define a functional on the automorphisms group Aut(M) on M by

$$\Phi(\tau) = (I - J)(\varphi_\tau)$$

for any $\varphi \in \mathcal{M}$, where φ_τ is defined as in (2.8) in Section 2 while ω_g is replaced by ω_{KE}. Then according to [BM], there is a $\sigma \in$ Aut(M) such that

$$\Phi(\sigma) = \inf_{\tau \in \mathrm{Aut(M)}} \Phi(\tau)$$

and consequently $\varphi_\sigma \in \Lambda_1^\perp(M, g_{KE})$. Note that φ_σ is invariant under a maximal compact subgroup of Aut(M) as same as φ. The inverse is also true. In fact, from the proof of uniqueness of Kähler-Einstein metrics in [BM], one can prove that $\varphi \in \Lambda_1^\perp(M, g_{KE})$ iff

$$(I - J)(\varphi) = \inf_{\tau \in \mathrm{Aut(M)}} \Phi(\tau).$$

Thus by Theorem 0.2 and the inequalities ([Au]),

$$\frac{1}{n} J(\varphi) \leq (I - J)(\varphi) \leq \frac{n-1}{n} J(\varphi), \ \forall \ \varphi \in \mathcal{M},$$

we get

$$\begin{aligned}
\mu(\varphi) &\geq \delta \inf_\tau I(\varphi_\tau) - C_\delta \\
&\geq \delta \inf_\tau (I - J)(\varphi_\tau) - C_\delta \\
&= \delta (I - J)(\varphi) - C_\delta \\
&\geq \frac{\delta}{2n} I(\varphi) - C_\delta.
\end{aligned}$$

Replacing $\frac{\delta}{2n}$ by δ, we obtain (4.13). □

Remark 4.1. *The existence of such an inequality (4.13) in Corollary (4.2) was conjectured by Tian for any Kähler-Einstein manifold ([T2]).*

5. θ_X on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$.

In this section, we estimate the quality θ_X on the manifolds $\mathbb{C}P^2\#1\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ and show that the conditions (4.3) and (4.12) in Section 4 are true for these two manifolds.

Proposition 5.1. *On $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ or $\mathbb{C}P^2\#1\overline{\mathbb{C}P^2}$, we have*

$$-2<\theta_X<1.$$

Proof. First we consider $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. In this case, we choose a Kähler form

$$\omega_g=\sqrt{-1}\partial\bar{\partial}\varphi\ \in\ 2\pi c_1(M)$$

which is given by a convex function,

$$\varphi=\frac{1}{2}\left[log(1+e^{2y_1}+e^{2y_2})+log(1+e^{2y_1})+log(1+e^{2y_2})-2y_1-2y_2\right].$$

Then M associates to a pentagon P_0 in $\mathbb{R}^2$ by using the moment map $x_i=\frac{\partial\varphi}{\partial y_i}$. Namely,

$$\begin{cases}\frac{\partial\varphi}{\partial y_1}=\frac{e^{2y_1}}{(1+e^{2y_1}+e^{2y_2})}+\frac{e^{2y_1}}{(1+e^{2y_1})}-1\\ \frac{\partial\varphi}{\partial y_2}=\frac{e^{2y_2}}{(1+e^{2y_1}+e^{2y_2})}+\frac{e^{2y_2}}{(1+e^{2y_2})}-1.\end{cases}$$

Thus P_0 is decided around by five edges:

$$x_1=1,\ x_2=1,\ x_1=-1,\ x_2=-1,\ x_1+x_2=1.$$

For any holomorphic vector field v on M, we normalize the potential function θ_v of v by

$$\int_M\theta_v e^{h_g}\omega_g^n=0,$$

where h_g is a potential function of the Ricci form of ω_g. Then θ_v satisfies (see [TZ]),

$$\theta_v=-\triangle\theta_v-v(h_g).$$

So the Futaki invariant can be computed by

$$F(v)=\int_M v(h_g)\omega^n=-\int_M\theta_v\omega^n=-(2\pi)^2\int_{P_0}\theta_v dx.$$

In particular,

$$\begin{aligned}F_1=F(\frac{\partial}{\partial y_1})&=-(2\pi)^2\int_{P_0}x_1dx_1dx_2\\&=-\frac{(2\pi)^2}{3},\end{aligned}\tag{5.1}$$

and

$$F_2=F(\frac{\partial}{\partial y_2})=-\frac{(2\pi)^2}{3},\tag{5.2}$$

where (y_1,y_2) is an affine coordinates system.

By using a translation $x_i' = x_i + \frac{1}{3vol(P)}$, we get a new pentagon P_1 so that

$$\int_{P_1} x_i' dx_1' dx_2' = 0, \;\; i = 1, 2.$$

Then θ_X associated to the extremal holomorphic vector field X is a from of

$$\theta_X = \theta_1 x_1' + \theta_2 x_2',$$

where θ_1 and θ_2 are two constants.

Since the Futaki invariant in the system (x_1', x_2') is computed by ([FM]),

$$\begin{aligned} F(v) &= \int_M v(h_g)\omega_g^n = -\int_M \theta_v \triangle h_g \omega_g^n = -\int_M \theta_v \theta_X \omega_g^n \\ &= -(2\pi)^2 \int_{P_1} \theta_v \theta_X dy, \end{aligned}$$

we get from (5.1) and (5.2),

$$\begin{cases} \left(\int_{P_1} x_1'^2 dx_1' dx_2'\right)\theta_1 + \left(\int_{P_1} x_1' x_2' dx_1' dx_2'\right)\theta_2 = \frac{1}{3} \\ \left(\int_{P_1} x_1' x_2' dx_1' dx_2'\right)\theta_1 + \left(\int_{P_1} x_2'^2 dx_1' dx_2'\right)\theta_2 = \frac{1}{3}. \end{cases}$$

A simple computation shows

$$\theta_1 = \theta_2 = -\frac{168}{409}.$$

Thus $-2 < \theta_X < 1$.

In the case of $\mathbb{C}P^2 \# 1\overline{\mathbb{C}P^2}$, we choose a Kähler form

$$\omega_g = \sqrt{-1}\partial\bar{\partial}\varphi \;\in\; 2\pi c_1(M)$$

which is given by a convex function,

$$\varphi = \frac{1}{2}\left[2log(1 + e^{2y_1} + e^{2y_2}) + log(e^{2y_1} + e^{2y_2}) - 2y_1 - 2y_2\right].$$

So M associates to a quadrilateral P_0 in $\mathbb{R}^2$ decided around by four edges:

$$x_1 + x_2 = -1, \;\; x_1 = -1, \;\; x_2 = -1, \;\; x_1 + x_2 = 1.$$

It can be computed in the similar way that

$$\theta_X = \frac{5}{29}x_1' + \frac{5}{29}x_2'$$

under the coordinates $\{x_1', x_2'\}$ satisfying

$$\int_{P_1} x_i' dx_1' dx_2' = 0, \;\; i = 1, 2.$$

Thus $-2 < \theta_X < 1$. $\square$

6. Kähler classes satisfying (0.2") on $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$.

In this section, we construct a kind of Kähler classes on the toric manifold $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, which satisfy the condition (0.2") in Introduction. It is well known that $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ with an anti-cannonical line bundle is corresponding to an integral polytope in $\mathbb{R}^2$,

$$P_0 = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 \le 1, -x_2 \le 1, -x_1 - x_2 \le 1, \\ -x_1 \le 1, x_2 \le 1, x_1 + x_2 \le 1\}.$$

Let

$$l_1 : x_1 = 1;$$
$$l_2 : -x_2 = 1;$$
$$l_3 : -x_1 - x_2 = 1;$$
$$l_4 : -x_1 = 1;$$
$$l_5 : x_2 = 1;$$
$$l_6 : x_1 + x_2 = 1;$$

be six edges of P_0, and D_i be six divisors on $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ corresponding to l_i. Then the cohomology class of a Kähler class is given by

$$\sum_{i=1}^{6} \lambda_i [D_i],$$

where λ_i are all positive numbers. So the Kähler class is corresponding to a convex polytope

$$P = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 \le \lambda_1, -x_2 \le \lambda_2, -x_1 - x_2 \le \lambda_3, \\ -x_1 \le \lambda_4, x_2 \le \lambda_5, x_1 + x_2 \le \lambda_6\}.$$

Moreover the Kähler class is integral when λ_i are all positive integers.

Lemma 6.1. *Let λ and μ be two positive numbers with $\frac{\lambda}{2} < \mu < 2\lambda$. Let*

$$\lambda_1 = \lambda_3 = \lambda_5 = \lambda,\ \lambda_2 = \lambda_4 = \lambda_6 = \mu.$$

Then the Futaki invariant $F(.)$ vanishes on the Kähler class associated to P.

Proof. By a direct computation, one sees

$$\int_P x_i dx = 0 \text{ and } \int_{\partial P} x_i dx = 0,\ i = 1, 2.$$

Then

$$F(\frac{\partial}{\partial y_i}) = 2^n n!(2\pi)^n (\int_{\partial P} x_i - \int_P \overline{R} x_i dx) = 0,\ i = 1, 2.$$

Thus the Futaki invariant vanishes. □

Lemma 6.2. *Let λ and μ, and $\lambda_i,\ i=1,...,6$ be numbers as in Lemma 6.1. Suppose*

$$\frac{\lambda}{1+\frac{\sqrt{10}}{5}} \le \mu \le (1+\frac{\sqrt{10}}{5})\lambda. \tag{6.1}$$

Then condition (0.2") is satisfied on the Kähler class associated to P .

Proof. By (1.9) in Section 1, one sees

$$\bar{R} = \frac{2(\mu+\lambda)}{4\lambda\mu-\mu^2-\lambda^2}.$$

Then condition (0.2") is equivalent to

$$\frac{2(\mu+\lambda)}{4\lambda\mu-\mu^2-\lambda^2} \le \frac{3}{\max\{\lambda,\mu\}}.$$

The late is equivalent to (6.1). □

Combining Lemma 6.1 and 6.2 to Theorem 0.2, we have

Proposition 6.1. *Let λ and μ be two positive numbers which satisfy (6.1). Let*

$$\lambda_1=\lambda_3=\lambda_5=\lambda,\ \lambda_2=\lambda_4=\lambda_6=\mu.$$

Then on the Kähler class with cohomology class

$$\lambda([D_1]+[D_3]+[D_5])+\mu([D_2]+[D_4]+[D_6]),$$

the K-energy $\mu(\varphi)$ is proper associated to subgroup T in the space of G_0-invariant Kähler metrics on $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ associated to P.

It is clear that for a pair $(\lambda,\mu)=(2,3)$ or $(\lambda,\mu)=(3,2)$, (6.1) is satisfied. Then by Proposition 6.1, K-energy $\mu(\varphi)$ is proper on the Kähler class on $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ associated to the integral polytope P with

$$\lambda_1=\lambda_3=\lambda_5=2,\ \lambda_2=\lambda_4=\lambda_6=3,$$

or

$$\lambda_1=\lambda_3=\lambda_5=3,\ \lambda_2=\lambda_4=\lambda_6=2.$$

It is interesting to study whether there exists a Kähler metric with constant scalar curvature on $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ in these two Kähler classes.

References

[**Ab**] Abreu, M., Kähler geometry of toric varieties and extremal metrics, Inter. J. Math. **9** (1998), 641-651.

[**Au**] Aubin, T., Réduction du cas positif de I'équation de Monge-Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité, J. Funct. Anal. **57** (1984), 143-153.

[**BM**] Bando, S & Mabuchi, T., Uniqueness of Einstein Kähler metrics modulo connected group actions, Algebraic geometry, Adv. Studies in Pure math, **10** (1987).

[**Ca**] Calabi, E., Extremal Kähler metrics. Seminar on differential geometry, Ann. of Math Stud., Princeton Univ. Press, Princeton, N.J. **102** (1982), 259-290.

[**D1**] Donaldson, S.K., Scalar curvature and projective embeddings, I, J. Diff. Geom. **59** (2001), 479-522.

[**D2**] Donaldson, S.K., Scalar curvature and stability of toric varieties, J. Diff. Geom. **62** (2002), 289-349.

[**FM**] Futaki, A. & Mabuchi, T., Bilinear forms and extremal Kähler vector fields associated with Kähler class, Math. Ann., **301** (1995), 199-210..

[**Gua**] Guan, D., On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundle, Math. Res. Lett. **6** (1999), 547-555.

[**Gui**] Guillemin, V., Monent maps and combinatorial invariants of Hamiltonian T^n-spaces, Progress in Math., Birkhäuser, Boston, Basel, Berlin, 1994.

[**LS**] Lebrun, C. & Simanca, S, Extremal Kähler metrics and complex deformation theory, Geom. Funct. Ana. **4** (1994), 298-336.

[**M1**] Mabuchi, T., A theorem of Calabi-Matsushima's type, Osaka J. Math. **39** (2002), 49-57.

[**M2**] Mabuchi, T., An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds, I, Invent. Math. **159** (2005), 225-243.

[**M3**] Mabuchi, T., An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds, II , Preprint, 2004.

[**Od**] Oda, T., Convex bodies and algebraic geometry, springer-verlag , 1988.

[**Si**] Siu, S.T., The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group, Ann. of Math. **127** (1988), 585-627.

[**Sz**] Székelyhidi, G., extremal metrics and K-stability, preprint, 2004.

[**T1**] Tian, G., On Calabi's conjecture for complex surfaces with positive first Chern class, Invent. Math. **101** (1990), 101-172.

[**T2**] Tian, G., Kähler-Einstein metrics with positive scalar curvature, Invent. Math. **130** (1997), 1-39.

[**TZ**] Tian, G. & Zhu, X.H., A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, Comm. Math. Helv. **77** (2002), 297-325.

[**WZ**] Wang, X.J. and Zhu, X.H., Kähler-Ricci solitons on toric Fano manifolds with positive first Chern class, Adv. in Math. **188 (2004)**, 87-103.

[**Ya**] Yau, S.T., Open problem in geometry, Proc. of Symp. in Pure Math. **54** (1993), 1-28.

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